Results Math (2025) 80:111
Online First

https://doi.org/10.1007/s00025-025-02426-1

Results in Mathematics

A Result on Certain Sums of Element Orders in Finite Groups

Marius Tărnăuceanu

Abstract. Given a finite group G of order $p^n m$, where p is a prime and $p \nmid m$, we denote by $\psi_p(G)$ the sum of orders of p-parts of elements in G. In the current note, we prove that $\psi_p(G) \leq \psi_p(C_{p^n m})$, where $C_{p^n m}$ is the cyclic group of order $p^n m$, and the equality holds if and only if G is p-nilpotent of a particular type. A generalization of this result is also presented.

Mathematics Subject Classification. Primary 20D60; Secondary 20D15, 20F18.

Keywords. Element orders, Finite groups.

1. Introduction

Let G be a finite group. In 2009, H. Amiri, S.M. Jafarian Amiri and I.M. Isaacs introduced in their paper [1] the function

$$\psi(G) = \sum_{x \in G} o(x),$$

where $o(x)$ denotes the order of x in G. They proved the following basic theorem:

Theorem A. *If G is a finite group of order n, then $\psi(G) \leq \psi(C_n)$, and we have equality if and only if G is cyclic.*

Since then many authors have studied the properties of the function $\psi(G)$ and its relations with the structure of G (see e.g. [3–5, 8–10, 15]).

Note that Theorem A follows from the next result:

Published online: 20 May 2025 Birkhäuser

Theorem B. *If G is a finite group of order n, then there is a bijection $f : G \longrightarrow C_n$ such that $o(x)$ divides $o(f(x))$, for all $x \in G$.*

This has been formulated as a question by I.M. Isaacs (see Problem 18.1 in [14]) and proved for some particular groups by F. Ladisch [13] and M. Amiri and S.M. Jafarian Amiri [2]. A proof for arbitrary groups has been recently given by M. Amiri [6].

Next we assume that $|G| = p^n m$, where p is a prime and $p \nmid m$, and consider the function

$$\psi_p(G) = \sum_{x \in G} o(x_p),$$

where x_p is the p-part of x in G. We remark that

$$\psi_p(G) = \sum_{x \in G} o(x^m)$$

and in particular

$$\psi_p(C_{p^n m}) = m\psi(C_{p^n}).$$

Our main result is stated as follows.

Theorem 1.1. *If G is a finite group of order $p^n m$ with p prime and $p \nmid m$, then $\psi_p(G) \leq \psi_p(C_{p^n m})$, and we have equality if and only if $G \cong H \rtimes C_{p^n}$, where H is a normal subgroup of order m of G.*

Its proof will be given in Sect. 2, while in Sect. 3 we will present a generalization of this theorem. Also, we remark that there is no constant $c \in (0, 1)$ such that if $\psi_p(G) > c\psi_p(C_{p^n m})$, then G belongs to a significant class of groups, as it happens for the original function ψ.

For the proof of Theorem 1.1, we need two well-known theorems of Frobenius [7] and N. Iiyori and H. Yamaki [11].

Theorem C. *Let G be a finite group whose order is divisible by a number n. Then the number of solutions of the equation $x^n = 1$ in G is a multiple of n.*

Theorem D. *Let G be a finite group whose order is divisible by a number n. If the set of solutions of the equation $x^n = 1$ in G has exactly n elements, then it forms a subgroup of G.*

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [7].

2. Proof of Theorem 1.1

First of all, we observe that if G is a finite group of order n, then Theorem B implies the existence of a partition $(L_d(G))_{d|n}$ of G such that for every divisor d of n we have:

a) $|L_d(G)| = \varphi(d)$, where φ is the Euler totient function;
b) $x^d = 1, \forall\, x \in L_d(G)$.

Moreover, the bijection f in Theorem B maps the elements of $L_d(G)$ to the elements of order d of C_n.

We are now able to prove our main result.

Proof of Theorem 1.1. Under the above notation, it is easy to see that

$$o(x)|o(f(x)) \Rightarrow o(x_p)|o(f(x)_p), \forall\, x \in G,$$

which leads to

$$\begin{aligned}\psi_p(G) &= \sum_{x\in G} o(x_p)\\ &\leq \sum_{x\in G} o(f(x)_p)\\ &= \sum_{y\in C_{p^n m}} o(y_p)\\ &= \psi_p(C_{p^n m}).\end{aligned}$$

Assume now that $\psi_p(G) = \psi_p(C_{p^n m})$ and let $0 \leq i \leq n$ and $m_1|m$. Then every element $x \in L_{p^i m_1}(G)$ can be written as $x = x_{pi}x'_i$, where $o(x_{pi})|p^i$ and $o(x'_i)|m_1$. We get

$$\psi_p(G) = \sum_{i=0}^{n} \sum_{m_1|m} \sum_{x\in L_{p^i m_1}(G)} o(x_{pi})$$

and

$$\psi_p(C_{p^n m}) = \sum_{i=0}^{n} \sum_{m_1|m} \varphi(p^i m_1)p^i,$$

implying that $o(x_{pi}) = p^i$, $\forall\, x \in L_{p^i m_1}(G)$. Thus

$$o(x_{pn}) = p^n, \forall\, x \in L_{p^n m_1}(G),$$

and so the Sylow p-subgroups of G are cyclic. Moreover

$$\begin{aligned}|\{x \in G \mid x^m = 1\}| &= \sum_{m_1|m} |L_{p^0 m_1}(G)|\\ &= \sum_{m_1|m} \varphi(m_1)\\ &= m\end{aligned}$$

and Theorem D shows that $H = \{x \in G \mid x^m = 1\}$ is a subgroup of G. Clearly, H is normal in G and it follows that $G \cong H \rtimes C_{p^n}$, completing the proof. $\square$

We remark that an alternative way to prove the cyclicity of the Sylow p-subgroups of G is as follows. Since

$$\psi_p(G) = \psi_p(C_{p^n m}) = m\psi(C_{p^n}) = m\,\frac{p^{2n+1}+1}{p+1}\,,$$

there is $x \in G$ such that

$$o(x_p) \geq \frac{p^{2n+1}+1}{p^n(p+1)} > p^{n-1},$$

i.e. $o(x_p) = p^n$.

We end this section with the following example.

Example. The group $G = \mathrm{SL}_2(\mathbb{F}_3) = Q_8 \rtimes C_9 = \mathit{SmallGroup}(72,3)$ has 1 element of order 1, 1 element of order 2, 2 elements of order 3, 6 elements of order 4, 2 elements of order 6, 24 elements of order 9, 12 elements of order 12 and 24 elements of order 18. The sets $(L_d(G))_{d|12}$ can be chosen as follows

- $L_1(G)$: 1 element of order 1,
- $L_2(G)$: 1 element of order 2,
- $L_3(G)$: 2 elements of order 3,
- $L_4(G)$: 2 elements of order 4,
- $L_6(G)$: 2 elements of order 6,
- $L_8(G)$: 4 elements of order 4,
- $L_9(G)$: 6 elements of order 9,
- $L_{12}(G)$: 4 elements of order 12,
- $L_{18}(G)$: 6 elements of order 9,
- $L_{24}(G)$: 8 elements of order 12,
- $L_{36}(G)$: 12 elements of order 9,
- $L_{72}(G)$: 24 elements of order 18,

and we easily get

$$\psi_2(G) = 387 = \psi_2(C_{72}) \text{ and } \psi_3(G) = 488 = \psi_3(C_{72}).$$

3. A Generalization

Let G be a finite group of order n and π be a set of primes contained in $\pi(n)$. Then every element x of G can be uniquely written as $x = x_\pi x_{\pi'}$, where $o(x_\pi)$ is a π-number and $o(x_{\pi'})$ is a π'-number. Define

$$\psi_\pi(G) = \sum_{x\in G} o(x_\pi).$$

Similarly with Theorem 1.1, one can prove the next result.

Theorem 3.1. *Under the above notation, we have $\psi_\pi(G) \leq \psi_\pi(C_n)$, and the equality holds if and only if $G \cong H \rtimes C$, where H is a normal subgroup of G of order the π'-part of n and C is a cyclic subgroup of G of order the π-part of n.*

Note that an immediate consequence of Theorem 3.1 is the following.

Corollary 3.2. *Under the above notation, if* $\psi_\pi(G) = \psi_\pi(C_n)$ *and* $\psi_{\pi'}(G) = \psi_{\pi'}(C_n)$*, then* $G \cong C_n$*.*

Finally, we formulate the following question concerning Theorem D in Sect. 1.

Question. Let G be a finite group of order n, and let $P \in Syl_p(G)$ be a cyclic group. If $|\{x \in G : x^{n_{p'}} = 1\}| = n_{p'}$, then $\{x \in G : x^{n_{p'}} = 1\}$ is a subgroup of G.

Acknowledgements

The author is grateful to the reviewer for remarks which improve the previous version of the paper.

Funding The author did not receive support from any organization for the submitted work.

Data Availability Statement My manuscript has no associated data.

Declarations

Conflict of Interest The author declares that he has no conflict of interest.

References

[1] Amiri, H., Jafarian Amiri, S.M., I.M.: Isaacs, Sums of element orders in finite groups, Comm. Algebra **37**, 2978–2980 (2009)

[2] Amiri, M., Jafarian Amiri, S.M.: Characterization of finite groups by a bijection with a divisible property on the element orders. Comm. Algebra **45**(8), 3396–3401 (2017)

[3] Amiri, H., Jafarian Amiri, S.M.: Sums of element orders on finite groups of the same order. J. Algebra Appl. **10**(2), 187–190 (2011)

[4] Jafarian Amiri, S.M.: Second maximum sum of element orders on finite nilpotent groups. Comm. Algebra **41**(6), 2055–2059 (2013)

[5] Jafarian Amiri, S.M., Amiri, M.: Second maximum sum of element orders on finite groups. J. Pure Appl. Algebra **218**(3), 531–539 (2014)

[6] Amiri, M.: *On a bijection between a finite group and cyclic group*, J. Pure Appl. Algebra **228** (7) (2024) ID 107632

[7] Frobenius, F.G.: *Verallgemeinerung des Sylowschen Satze*, Berliner Sitz. (1895), 981-993

[8] Herzog, M., Longobardi, P., Maj, M.: An exact upper bound for sums of element orders in non-cyclic finite groups. J. Pure Appl. Algebra **222**(7), 1628–1642 (2018)

[9] Herzog, M., Longobardi, P., Maj, M.: Sums of element orders in groups of order $2m$ with m odd. Comm. Algebra **47**(5), 2035–2048 (2019)

[10] Herzog, M., Longobardi, P., Maj, M.: *The second maximal groups with respect to the sum of element orders*, J. Pure Appl. Algebra **225** (3) (2020), article ID 106531

[11] Iiyori, N., Yamaki, H.: On a conjecture of Frobenius. Bull. Amer. Math. Soc. **25**, 413–416 (1991)

[12] Isaacs, I.M.: Finite group theory. Amer. Math. Soc, Providence, R.I. (2008)

[13] Ladisch, F.: *Order-increasing bijection from arbitrary groups to cyclic groups*, http://mathoverflow.net/a/107395

[14] Mazurov, V.D., Khukhro, E.I.:*The Kourovka Notebook. Unsolved Problems in Group Theory*, 18th ed., Institute of Mathematics, Russian Academy of Sciences, Siberrian Division, Novosibirsk, arXiv:1401.0300v25, (2014)

[15] Shen, R., Chen, G., Wu, C.: On groups with the second largest value of the sum of element orders. Comm. Algebra **43**(6), 2618–2631 (2015)

Marius Tărnăuceanu
Faculty of Mathematics
"Al.I. Cuza" University
Iaşi
Romania
e-mail: tarnauc@uaic.ro

Received: November 3, 2024.
Accepted: April 20, 2025.